

Growth of p th means of analytic and subharmonic functions in the unit disc and angular distribution of zeros

Igor Chyzhykov

September 22, 2015

In memory of Professor Anatolii Grishin

Abstract

Answering a question of A.Zygmund in [22] G.MacLane and L.Rubel described boundedness of L_2 -norm w.r.t. the argument of $\log |B|$, where B is a Blaschke product. We generalize their results in several directions. We describe growth of p th means, $p \in (1, \infty)$, of subharmonic functions bounded from above in the unit disc. Necessary and sufficient conditions are formulated in terms of the complete measure (of a subharmonic function) in the sense of A.Grishin. We also prove sharp estimates of the growth of p th means of analytic and subharmonic functions of finite order in the unit disc.

1 Introduction and main results

1.1 Some results on growth and angular distribution of zeros of Blaschke products

In the present paper we investigate an interplay between zero distribution and growth of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Especially we are interested in growth of logarithmic means.

Given a sequence (a_n) in \mathbb{D} such that $\sum_n (1 - |a_n|) < \infty$, we consider the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{a_n}(a_n - z)}{|a_n|(1 - z\overline{a_n})}. \quad (1)$$

It was A. Zygmund (see [22]) who asked to describe those sequences (a_n) in \mathbb{D} that

$$I(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log |B(re^{i\theta})|)^2 d\theta$$

is bounded. In [22] G. Maclane and L. Rubel answered this question using Fourier series method.

Theorem A ([22, Theorem 1]). *A necessary and sufficient condition that $I(r)$ be bounded is that $J(r)$ be bounded, where*

$$J(r) = \sum_{k=1}^{\infty} \frac{1}{k^2} \left| (r^k - r^{-k}) \sum_{|a_n| \leq r} \bar{a}_n^k + r^k \sum_{|a_n| > r} (\bar{a}_n^k - a_n^{-k}) \right|^2.$$

Since it was difficult to check boundedness of $J(r)$ they gave also the following sufficient condition.

Let $n(r, B)$ be the number of zeros in the closed disc $\overline{D}(0, r)$; here and in what follows $D(z, r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\}$.

Theorem B ([22]). *If*

$$n(r, B) = O((1 - r)^{-\frac{1}{2}}), \quad r \in (0, 1), \quad (2)$$

then $I(r)$ is bounded.

They also noted that (2) is equivalent to the condition

$$\sum_{|a_n| > r} (1 - |a_n|) = O(\sqrt{1 - r}).$$

MacLane and Rubel also proved that (2) becomes necessary if all zeros lie on a finitely many rays emanating the origin, but it is not the case in general. After that C. N. Linden ([19, Corollary 1]) generalized this showing that it is sufficient to require that the zero sequence is contained in a finite number of Stolz angles with vertices on $\partial\mathbb{D}$. The last assertion is a consequence of the following result.

Let

$$\mathcal{R}(re^{i\varphi}, \sigma) = \left\{ \zeta : r \leq |\zeta| \leq \frac{1+r}{2}, |\arg \zeta - \varphi| \leq \sigma \right\}.$$

Theorem C ([19, Theorem 1]). *If $I(r) < M$, $0 < r < 1$. Then*

$$\#\{a_n \in \mathcal{R}(re^{i\varphi}, \varkappa(1-r)^\gamma)\} \leq \begin{cases} \frac{C\sqrt{M}(1+\sqrt{\varkappa})}{r(1-r)^{\frac{1}{2}}}, & \gamma \geq 1, \\ \frac{C\sqrt{M}(1+\sqrt{\varkappa})}{r(1-r)^{1-\frac{\gamma}{2}}}, & 0 \leq \gamma < 1. \end{cases} \quad (3)$$

Results of MacLane and Rubel show that the order of magnitude of the first estimate (3) is the best possible. Linden ([19]) also established sharpness of the estimate for $\gamma \in [0, 1)$.

The main growth characteristic which will be studied here is p th integral mean of $\log |f|$, where f is analytic in \mathbb{D} . Since our approach is natural for subharmonic functions we introduce means for the class of subharmonic functions in \mathbb{D} . Note that $\log |f|$ is subharmonic provided that f is analytic. Characterization of zeros of analytic functions f with $\log |f| \in L^p(\mathbb{D})$ is obtained in [2].

For a subharmonic function u in \mathbb{D} and $p \geq 1$ we define

$$m_p(r, u) = \left(\frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}, \quad 0 < r < 1,$$

$$\rho_p[u] = \limsup_{r \uparrow 1} \frac{\log^+ m_p(r, u)}{-\log(1-r)}.$$

The growth of $m_p(r, \log |f|)$ was studied in many papers, for instance [20], [21], [23], [26], [12], [27], [3], [7]. Nevertheless, to the best of our knowledge, only one paper, namely [23], contains criteria of boundedness of p th means when $u = \log |B|$. Unfortunately, proofs have not been published yet.

In [23] Ya.V. Mykytyuk and Ya.V. Vasylykiv introduced two auxiliary functions defined on $\partial\mathbb{D}$ by (a_n) :

$$\psi_r(\zeta) = \sum_{r \leq |a_n| < 1} \frac{(1 - |a_n|)^2}{|\zeta - a_n|^2}, \quad \zeta \in \partial\mathbb{D}, r \in [0, 1).$$

and $\varphi(\zeta)$, which satisfies the relation

$$\varphi(\zeta) \asymp \Phi(\zeta) := \#\{a_n : |1 - a_n \bar{\zeta}| < 2(1 - |a_n|)\},$$

i.e. the number of zeros in the Stolz angle with the vertex ζ . They established that ψ_0 and Φ belong to the same classes $L^p(\partial\mathbb{D})$, $p \in [1, \infty)$, and $\psi_0 \log |\psi_0|$ and $\Phi \log |\Phi|$ belong to $L^1(\partial\mathbb{D})$, simultaneously. Moreover, for a branch of $\log B$ in \mathbb{D} with the radial cuts $[a_k, \frac{a_k}{|a_k|})$ the following statement holds.

Theorem D ([23]). *Let B be a Blaschke product, and $p \in (1, \infty)$. Then:*

- 1) $m_p(r, \log B)$ is bounded on $[0, 1)$ if and only if $\psi_0 \in L^p(\partial\mathbb{D})$.
- 2) $m_1(r, \log B)$ is bounded if and only if $\psi_0 \log^+ \psi_0 \in L^1(\partial\mathbb{D})$.
- 3) $m_p(r, \log |B|)$ is bounded on $[0, 1)$ if and only if

$$\sup_{0 < r < 1} \int_0^{2\pi} \left(\int_0^{2\pi} \frac{1 - r^2}{|re^{i\theta} - e^{i\varphi}|^2} \psi_r(e^{i\theta}) d\theta \right)^p d\varphi < \infty.$$

$$4) \psi_0 \in L^p(\partial\mathbb{D}) \Rightarrow \sup_{0 < r < 1} m_p(r, \log |B|) < \infty.$$

$$5) n(r, f) = O((1 - r)^{-\frac{1}{p}}) \Rightarrow \sup_{0 < r < 1} m_p(r, \log |B|) < \infty.$$

Relations between conditions on the zeros of a Blaschke product B and the belongness of $\arg B(e^{i\theta})$ to L^p spaces $0 < p \leq \infty$ were investigated by A. Rybkin ([24]).

The following tasks arise naturally:

- i) Describe the growth of p th means of $\log |f|$ where f is a bounded analytic function in \mathbb{D} , $1 < p < \infty$.
- ii) Find more ‘explicit’ conditions on zero distribution than that of Theorems A and D.
- iii) Extend the description on functions of finite order of growth.
- iv) Find simple conditions that provide a prescribed growth of $m_p(r, \log |f|)$.

In the paper we accomplish these tasks.

1.2 Complete measure and main results

Our method is based on a concept of so the called *complete measure* of a subharmonic function introduced by A. Grishin in the case of the half-plane (see [14], [15]). As it was mentioned there, this concept allows to obtain very simple representation for a subharmonic function of finite order and defines this function up to a harmonic addend in the closure of the domain.

Let SH^∞ be the class of subharmonic functions in \mathbb{D} bounded from above. In particular, $\log |f| \in SH^\infty$ if $f \in H^\infty$, the space of bounded analytic functions in \mathbb{D} . In this case (cf. [17, Ch.3.7])

$$u(z) = \int_{\mathbb{D}} \log \frac{|z - \zeta|}{|1 - z\bar{\zeta}|} d\mu_u(\zeta) - \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} d\psi(\zeta) + C. \quad (4)$$

where ψ is a positive Borel measure, μ_u is the Riesz measure of u ([17]), and $\int_{\mathbb{D}} (1 - |\zeta|) d\mu_u(\zeta) < \infty$. The *complete measure* λ_u of u in the sense of Grishin is defined [14, 15] by the boundary measure and the Riesz measure of $u(z)$. But, since [8]

$$\lim_{r \uparrow 1} \int_{\theta_1}^{\theta_2} \int_{\mathbb{D}} \log \frac{|re^{i\theta} - \zeta|}{|1 - re^{i\theta}\bar{\zeta}|} d\mu_u(\zeta) d\theta = 0, \quad -\pi \leq \theta_1 < \theta_2 \leq \pi,$$

i.e. the boundary values of the first integral from (4) do not contribute to the boundary measure, we can define λ_u of a Borel set $M \subset \overline{\mathbb{D}}$ such that $M \cap \partial\mathbb{D}$ is measurable with respect to Lebesgue measure on $\partial\mathbb{D}$ by

$$\lambda_u(M) = \int_{\mathbb{D} \cap M} (1 - |\zeta|) d\mu_u(\zeta) + \psi(M \cap \partial\mathbb{D}). \quad (5)$$

The measure $\lambda = \lambda_u$ has the following properties:

- (1) λ is finite on $\overline{\mathbb{D}}$;
- (2) λ is non-negative;
- (3) λ is a zero measure outside $\overline{\mathbb{D}}$;
- (4) $d\lambda|_{\partial\mathbb{D}}(\zeta) = d\psi(\zeta)$;
- (5) $d\lambda|_{\mathbb{D}}(\zeta) = (1 - |\zeta|) d\mu_u(\zeta)$.

If B is a Blaschke product of form (1), then $\lambda_{\log |B|}(M) = \sum_{a_n \in M} (1 - |a_n|)$.
Let

$$\mathcal{C}(\varphi, \delta) = \{\zeta \in \overline{\mathbb{D}} : |\zeta| \geq 1 - \delta, |\arg \zeta - \varphi| \leq \pi\delta\}$$

be the Carleson box based on the arc $[e^{i(\varphi - \pi\delta)}, e^{i(\varphi + \pi\delta)}]$.

The following theorem describes the growth of integral means for $u \in SH^\infty$.

Theorem 1. *Let $u \in SH^\infty$, $\gamma \in (0, 1]$, $p \in (1, \infty)$. Let λ be the complete measure of u . Necessary and sufficient that*

$$m_p(r, u) = O((1 - r)^{\gamma-1}), \quad r \uparrow 1, \quad (6)$$

hold is that

$$\left(\int_0^{2\pi} \lambda^p(\mathcal{C}(\varphi, \delta)) d\varphi \right)^{\frac{1}{p}} = O(\delta^\gamma), \quad 0 < \delta < 1. \quad (7)$$

Theorem 2. Let $f \in H^\infty$, $\gamma \in (0, 1]$, $p \in (1, \infty)$. Let λ be the complete measure of $\log |f|$. Necessary and sufficient that

$$m_p(r, \log |f|) = O((1-r)^{\gamma-1}), \quad r \uparrow 1, \quad (8)$$

hold is that (7).

Remark. It was proved in [3] that if $\text{supp } \lambda \subset \partial\mathbb{D}$, i.e., u is harmonic, $\gamma \in (0, 1)$ then (7) is equivalent to (10).

Remark. Though Theorems 1 and 2 look like Carleson type results we cannot use standard tools (e.g. [11, Chap.9]) here, because u and $\log |f|$ have logarithmic singularities.

The crucial role in the proof of sufficiency plays Lemma 1. In order to prove necessity of Theorems 1 and 2 we essentially use the fact that kernels in representation (4) preserve the sign. The method allows to spread the sufficient part of Theorems 1 and 2 to functions of finite order of growth (see Theorems 4, 5 below).

Under additional assumptions on zero location of a Blaschke product (support of the Riesz measure of the Green potential) (??) could be simplified.

Theorem 3. Let

$$u(z) = \int_{\mathbb{D}} \log \frac{|z - \zeta|}{|1 - z\bar{\zeta}|} d\mu_u(\zeta), \quad \int_{\mathbb{D}} (1 - |\zeta|) d\mu_u(\zeta) < \infty, \quad (9)$$

$\alpha \in [0, 1)$, $p \in (1, \infty)$, $\alpha + \frac{1}{p}$. Suppose that $\text{supp } \mu_u$ is contained in a finite number of Stolz angles with vertices on $\partial\mathbb{D}$. Necessary and sufficient that

$$m_p(r, u) = O((1-r)^{-\alpha}), \quad r \uparrow 1, \quad (10)$$

hold is that

$$n(r, u) := \mu_u(\overline{D(0, r)}) = O((1-r)^{-\alpha - \frac{1}{p}}), \quad r \uparrow 1. \quad (11)$$

Similar to the case $\alpha = 0$, the growth condition (11) is appeared to be sufficient for (10) when u is of finite order (see Theorem 6 below).

Remark. Taking $u = \log |B|$, we obtain a generalization of MacLane and Rubel, and Linden's results mentioned in Subsection 1.1.

Remark. If $\alpha + \frac{1}{p} \geq 1$ correlations (10) and (11) become trivial, see remark after Theorem 5.

1.3 Growth and zero distribution of zeros of finite order

In order to formulate results on angular distribution for unbounded analytic functions we need some growth characteristics. The standard characteristics are the maximum modulus $M(r, f) = \max\{|f(z)| : |z| = r\}$, and the Nevanlinna characteristic ([16]) $T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$, $x^+ = \max\{x, 0\}$. Note that both of them are bounded for $f = B$. Note that the order defined by $T(r, f)$ coincides with $\rho_1[\log |f|]$.

It follows from results of C.Linden [18] that $M(r, f)$ does take into account the angular distribution of the zeros when it grows sufficiently fast, namely, when the order of growth

$$\rho_M[f] = \limsup_{r \uparrow 1} \frac{\log^+ \log^+ M(r, f)}{-\log(1-r)} \geq 1.$$

To be more precise, consider the canonical product

$$\mathcal{P}(z, (a_k), s) := \prod_{k=1}^{\infty} E(A(z, a_k), s),$$

where

$$E(w, s) = (1 - w) \exp\{w + w^2/2 + \cdots + w^s/s\}, \quad s \in \mathbb{Z}_+,$$

is the Weierstrass primary factor, $A(z, \zeta) = \frac{1 - |\zeta|^2}{1 - z\bar{\zeta}}$, $z \in \mathbb{D}$, $\zeta \in \overline{\mathbb{D}}$.

Let

$$\tilde{\square}(re^{i\varphi}) := \mathcal{R}\left(re^{i\varphi}, \frac{1-r}{2}\right),$$

$\nu(re^{i\varphi})$ be the number of zeros of \mathcal{P} in $\tilde{\square}(re^{i\varphi})$. We define

$$\nu_1(\varphi) = \limsup_{r \uparrow 1} \frac{\log^+ \nu(re^{i\varphi}, \mathcal{P})}{-\ln(1-r)}, \quad \nu[\mathcal{P}] = \sup_{\varphi} \nu_1(\varphi). \quad (12)$$

With the notation above we have ([18, Theorem V])

$$\rho_M[\mathcal{P}] \begin{cases} = \nu[\mathcal{P}], & \rho_M[\mathcal{P}] \geq 1, \\ \leq \nu[\mathcal{P}] \leq 1, & \rho_M[\mathcal{P}] < 1. \end{cases} \quad (13)$$

Given a Borel measure μ on \mathbb{D} satisfying

$$\int_{\mathbb{D}} (1 - |\zeta|)^{s+1} d\mu_f(\zeta) < \infty, \quad s \in \mathbb{N} \cup \{0\}, \quad (14)$$

define the canonical integral as

$$U(z; \mu, s) := \int_{\mathbb{D}} \log |E(A(z, \zeta), s)| d\mu(\zeta). \quad (15)$$

Let ($q > -1$)

$$S_\alpha(z) = \Gamma(1+q) \left(\frac{2}{(1-z)^{q+1}} - 1 \right), \quad P_q(z) = \operatorname{Re} S_q(z), \quad S_q(0) = \Gamma(q+1).$$

Note that S_0 and P_0 are the Schwarz and Poisson kernels, respectively.

Let u be a subharmonic function in \mathbb{D} of the form

$$u(z) = U(z; \mu, s) - \frac{1}{2\pi} \int_0^{2\pi} P_s(ze^{-i\theta}) d\psi^*(\theta) + C, \quad (16)$$

where $\psi^* \in BV[0, 2\pi]$, μ is the Riesz measure of u satisfying (14). Note that every subharmonic function u of finite order in \mathbb{D} , i.e. satisfying $\log \max\{u(z) : |z| = r\} = O(\log \frac{1}{1-r})$ ($r \uparrow 1$), can be represented in the form (16) for an appropriate $s \in \mathbb{N} \cup \{0\}$ ([17], [9, Chap.9]).

Let M be Borel's subset of \mathbb{D} such that $M \cap \partial\mathbb{D}$ is measurable with respect to the Lebesgue measure on $\partial\mathbb{D}$. Let u be a subharmonic function in \mathbb{D} of the form (16). We set

$$\lambda_u(M) = \int_{\mathbb{D} \cap M} (1 - |\zeta|)^{s+1} d\mu_u(\zeta) + \psi(M \cap \partial\mathbb{D}), \quad (17)$$

where μ_u is the Riesz measure of u , ψ is the (signed) Stieltjes measure associated with ψ^* . Note that, in the case $u = \log |f|$ we have $\mu_{\log |f|}(\zeta) = \sum_n \delta(\zeta - a_n)$, where (a_n) is the zero sequence of f .

Let $|\lambda|$ denote the total variation of λ .

Theorem 4. *Let u be a subharmonic function in \mathbb{D} of the form (16), $\gamma \in (0, s+1]$, $p \in (1, \infty)$. Let λ be defined by (17). If*

$$\left(\int_0^{2\pi} |\lambda|^p(\mathcal{C}(\varphi, \delta)) d\varphi \right)^{\frac{1}{p}} = O(\delta^\gamma), \quad 0 < \delta < 1, \quad (18)$$

holds, then

$$m_p(r, u) = O((1-r)^{\gamma-s-1}), \quad r \uparrow 1. \quad (19)$$

Theorem 5. *Let f be of the form*

$$f(z) = C_q z^\nu \mathcal{P}(z, (a_k), q) \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} S_q(z e^{-i\theta}) d\psi^*(\theta) \right\}, \quad (20)$$

where $\psi^* \in BV[0, 2\pi]$, (a_k) is the zero sequence of f such that $\sum_k (1 - |a_k|)^{q+1} < +\infty$, $\nu \in \mathbb{Z}_+$, $C_q \in \mathbb{C}$. Let $\gamma \in (0, s+1]$, $p \in (1, \infty)$. Let λ be defined by (17) for $u = \log |f|$. If

$$\left(\int_0^{2\pi} |\lambda|^p(\mathcal{C}(\varphi, \delta)) d\varphi \right)^{\frac{1}{p}} = O(\delta^\gamma), \quad 0 < \delta < 1, \quad (21)$$

holds, then

$$m_p(r, \log |f|) = O((1-r)^{\gamma-s-1}), \quad r \uparrow 1. \quad (22)$$

Remark. Suppose that μ is 1-periodic measure on \mathbb{R} finite on the compact Borel sets, and $p \geq 1$. Then

$$\left(\int_0^1 \left(\mu((x-\delta, x+\delta)) \right)^p dx \right)^{\frac{1}{p}} = O(\delta^{\frac{1}{p}}), \quad \delta \in (0, 1).$$

In fact, assume that $\mu([0, 1)) = C$. Then by Fubini's theorem

$$\begin{aligned} \int_0^1 \left(\mu((x-\delta, x+\delta)) \right)^p dx &\leq (2C)^{p-1} \int_0^1 \mu((x-\delta, x+\delta)) dx = \\ &= (2C)^{p-1} \int_0^1 \int_{(x-\delta, x+\delta)} d\mu(y) dx \leq \\ &\leq (2C)^{p-1} \int_{-\delta}^{1+\delta} d\mu(y) \int_{(y-\delta, y+\delta)} dx \leq 2\delta \mu((-\delta, 1+\delta)) \leq 3(2C)^p \delta. \end{aligned}$$

It follows from representation (16) that $\rho_1[u] \leq s$. Then (19) implies

$$\rho_p[u] \leq s + 1 - \frac{1}{p}.$$

It is known that this is a sharp inequality ([20, 21]), in general. However, Theorems 4 and 5 characterize classes where $\rho_p[u]$ takes a particular value.

Examples in Section 4 show that the assertion of Theorem 5 is sharp.

The following theorem provides a sharp estimate for means of canonical integrals or products in terms of growth of their Riesz measures.

Theorem 6. Suppose that u is of the form (15), $s \in \mathbb{N} \cup \{0\}$, $p \in (1, \infty)$, and $\alpha > 0$ are such that $\alpha + \frac{1}{p} < s + 1$. If (11) holds, then (10) is valid.

2 Kernels $K_s(z, \zeta)$ and representation of functions of finite order

We define

$$K(z, \zeta) = \frac{G(z, \zeta)}{1 - |\zeta|} = \frac{1}{1 - |\zeta|} \log \left| \frac{1 - z\bar{\zeta}}{z - \zeta} \right|, \quad z \in \mathbb{D}, \zeta \in \mathbb{D}, z \neq \zeta,$$

where $G(z, \zeta)$ is the Green function for \mathbb{D} . We have the following properties of $K(z, \zeta)$, $z = re^{i\varphi}$, $\zeta = \rho e^{i\theta}$.

Proposition 1. The following hold:

- a) $K(z, 0) = -\log |z|$.
- b) $0 \leq K(z, \zeta) \leq \frac{1 - |z|^2}{|z - \zeta|^2}$.
- c) If $D \Subset \mathbb{D}$, then uniformly in $z \in D$

$$\lim_{\rho \uparrow 1} K(z, \rho e^{i\theta}) = \frac{1 - |z|^2}{|\rho e^{i\theta} - z|^2} = P_0(ze^{-i\theta}).$$

d)

$$|K(z, \zeta)| \geq \frac{1}{12} \frac{1 - |z|^2}{|z - \zeta|^2}, \quad 1 - |\zeta| \leq \frac{1}{2}(1 - |z|).$$

Proof of the proposition. b) We have

$$\begin{aligned} 0 \leq K(re^{i\varphi}, \rho e^{i\theta}) &= \frac{1}{2(1 - \rho)} \log \frac{1 - 2r\rho \cos(\varphi - \theta) + r^2\rho^2}{r^2 - 2r\rho \cos(\varphi - \theta) + \rho^2} = \\ &= \frac{1}{2(1 - \rho)} \log \left(1 + \frac{(1 - r^2)(1 - \rho^2)}{r^2 - 2r\rho \cos(\varphi - \theta) + \rho^2} \right) \leq \\ &\leq \frac{1}{2(1 - \rho)} \frac{(1 - r^2)(1 - \rho^2)}{r^2 - 2r\rho \cos(\varphi - \theta) + \rho^2} \leq \frac{1 - r^2}{|re^{i\varphi} - \rho e^{i\theta}|^2}. \end{aligned}$$

c) The assertion easily follows from the equality

$$K(z, \zeta) = \frac{1}{2(1 - |\zeta|)} \log \left(1 + \frac{(1 - |z|^2)(1 - |\zeta|^2)}{|z - \zeta|^2} \right)$$

see b).

d) It is proved in [7]. □

Due to d), we set $K(z, e^{i\theta}) := P_0(ze^{-i\theta})$ preserving continuity of K on $\partial\mathbb{D}$ with respect to the second variable.

Let $s \in \mathbb{N}$. We write

$$K_s(z, \zeta) = -\frac{\log |E(A(z, \zeta), s)|}{(1 - |\zeta|)^{s+1}}, \quad \zeta \in \overline{\mathbb{D}}, z \in \mathbb{D}, z \neq \zeta,$$

i.e. $K(z, \zeta) = K_0(z, \zeta)$, we set $K_s(z, z) = -\infty$, $z \in \mathbb{D}$.

Let $D^*(z, \sigma) = \{\zeta : \left| \frac{z - \zeta}{1 - \bar{z}\zeta} \right| < \sigma\}$ be the pseudohyperbolic disc with the center z and radius $\sigma \in (0, 1]$.

Proposition 2. *The following hold:*

i)

$$|K_s(z, \zeta)| \leq \frac{C(s)}{|1 - z\bar{\zeta}|^{s+1}}, \quad \zeta \notin D^*(z, \frac{1}{7}). \quad (23)$$

ii)

$$|K_s(z, \zeta)| \leq \frac{C}{(1 - |\zeta|)^{s+1}} \log \left| \frac{1 - z\bar{\zeta}}{z - \zeta} \right|, \quad \zeta \in D^*(z, \frac{1}{7}). \quad (24)$$

iii) If $z \in D \Subset \mathbb{D}$, then

$$K_s(z, \rho e^{i\theta}) \Rightarrow \frac{2^{s+1}}{s+1} \operatorname{Re} \frac{1}{(1 - ze^{-i\theta})^{s+1}} = \frac{2^s P_s(ze^{-i\theta})}{(s+1)!} + C(s), \quad \rho \uparrow 1.$$

Proof of the proposition. The upper estimate for $K_s(z, \zeta)$

$$K_s(z, \zeta) \leq \frac{2^{s+2}}{(1 - |\zeta|)^{s+1}} |A(z, \zeta)|^{s+1} \leq \frac{2^{2s+3}}{|1 - z\bar{\zeta}|^{s+1}}, \quad z \in \mathbb{D}, \zeta \in \overline{\mathbb{D}}, \quad (25)$$

follows from the known estimate of the primary factor ([28, Chap. V.10]). Also,

$$K_s(z, \zeta) = \frac{\operatorname{Re} \sum_{j=1}^{\infty} \frac{1}{j} (A(z, \zeta))^j}{(1 - |\zeta|)^{s+1}}, \quad (26)$$

provided that $|A(z, \zeta)| < \frac{1}{2}$, so

$$|K_s(z, \zeta)| \leq \frac{2|A(z, \zeta)|^{s+1}}{(s+1)(1 - |\zeta|)^{s+1}} \leq \frac{2^{s+2}}{s+1} \frac{1}{|1 - z\bar{\zeta}|^{s+1}}.$$

Hence, it remains to consider the case when $|A(z, \zeta)| \geq \frac{1}{2}$.

Since for all $z \in \mathbb{D}$, $\zeta \in \overline{\mathbb{D}}$ $|A(z, \zeta)| \leq 2$, we have for $\zeta \notin D^*(z, \frac{1}{7})$

$$\begin{aligned} K_s(z, \zeta)(1 - |\zeta|)^{s+1} &= \log \left| \frac{z - \zeta}{1 - z\bar{\zeta}} \right| + \operatorname{Re} \sum_{j=1}^s \frac{A(z, \zeta)^j}{j} \geq -\log 7 - \sum_{j=1}^s \frac{2^j}{j} = \\ &= -C(s) \geq -C(s)2^{s+1}|A(z, \zeta)|^{s+1}. \end{aligned}$$

Hence,

$$K_s(z, \zeta) \geq -\frac{C(s)}{|1 - z\bar{\zeta}|^{s+1}}, \quad \zeta \notin D^*(z, \frac{1}{7}), \quad (27)$$

and i) follows.

Similar arguments give ii).

Let us prove iii). If $z \in D \Subset \mathbb{D}$, then it follows from the representation (26) that

$$K_s(z, \rho e^{i\theta}) \Rightarrow \frac{2^{s+1}}{s+1} \operatorname{Re} \frac{1}{(1 - ze^{-i\theta})^{s+1}} = \frac{2^s P_s(ze^{-i\theta})}{(s+1)!} + C(s), \quad \rho \uparrow 1.$$

□

Due to properties of $K_s(z, \zeta)$ the representation (16) could be rewritten in the form (cf. [14], [13, Part II])

$$u(z) = - \int_{\mathbb{D}} K_s(z, \zeta) d\lambda(\zeta) + C, \quad (28)$$

where $\frac{2^s}{(s+1)!} d\lambda(e^{i\theta}) = \frac{1}{2\pi} d\psi^*(\theta)$, and $\lambda = \lambda_u$ is defined by (17).

Similarly, for any $u \in \text{SH}^\infty$ we have the following representation (cf. (4))

$$u(z) = - \int_{\mathbb{D}} K(z, \zeta) d\lambda_u(\zeta) + C. \quad (29)$$

Remark. The idea of such representation goes back to results of Martin [1, Chap. XIV].

3 Proofs

Sufficiency of Theorem 1. We write

$$u_1(z) = - \int_{D^*(z, \frac{1}{7})} K(z, \zeta) d\lambda(\zeta), \quad u_2(z) = - \int_{\mathbb{D} \setminus D^*(z, \frac{1}{7})} K(z, \zeta) d\lambda(\zeta).$$

Let us estimate $I_1 = \int_{-\pi}^{\pi} |u_1(re^{i\varphi})|^p d\varphi$.

By the Hölder inequality

$$\begin{aligned} |u_1(re^{i\varphi})| &= \int_{D^*(re^{i\varphi}, \frac{1}{7})} \log \left| \frac{1 - re^{i\varphi} \bar{\zeta}}{re^{i\varphi} - \zeta} \right| d\mu(\zeta) \leq \\ &\leq \left(\int_{D^*(re^{i\varphi}, \frac{1}{7})} \left(\log \left| \frac{1 - re^{i\varphi} \bar{\zeta}}{re^{i\varphi} - \zeta} \right| \right)^p d\mu(\zeta) \right)^{\frac{1}{p}} \left(\mu \left(D^*(re^{i\varphi}, \frac{1}{7}) \right) \right)^{\frac{p-1}{p}}, \end{aligned}$$

hence

$$I_1 \leq \int_{-\pi}^{\pi} \left(\int_{D^*(re^{i\varphi}, \frac{1}{7})} \left| \log \left| \frac{1 - re^{i\varphi} \bar{\zeta}}{re^{i\varphi} - \zeta} \right| \right|^p d\mu(\zeta) \mu^{p-1} \left(D^*(re^{i\varphi}, \frac{1}{7}) \right) \right) d\varphi.$$

Since $D^*(z, \frac{h}{2+h}) \subset D(z, (1-|z|)h)$ ([5]), with $h = \frac{1}{3}$ we get

$$D^*\left(z, \frac{1}{7}\right) \subset D\left(z, (1-|z|)\frac{1}{3}\right) \subset \square\left(z, \frac{1}{3}(1-|z|)\right),$$

where $\square(re^{i\varphi}, \sigma) = \{\rho e^{i\theta} : |\rho - r| \leq \sigma, |\theta - \varphi| \leq \sigma\}$. Therefore, using Fubini's

theorem, we deduce

$$\begin{aligned}
I_1 &\leq \int_{-\pi}^{\pi} \left(\int_{\square(z, \frac{1}{3}(1-r))} \left(\log \left| \frac{1 - re^{i\varphi} \bar{\zeta}}{re^{i\varphi} - \zeta} \right| \right)^p \right) \mu^{p-1} \left(\square(z, \frac{1}{3}(1-r)) \right) d\mu(\zeta) d\varphi \leq \\
&\leq \int_{-\pi}^{\pi} \left(\int_{\square(z, \frac{1}{3}(1-r))} \left(\log \left| \frac{1 - re^{i\varphi} \bar{\zeta}}{re^{i\varphi} - \zeta} \right| \right)^p \right) \mu^{p-1} \left(\square(re^{i \arg \zeta}, \frac{2}{3}(1-r)) \right) d\mu(\zeta) d\varphi = \\
&\leq \iint_{\substack{-\pi - \frac{1-r}{3} \leq \theta \leq \pi + \frac{1-r}{3} \\ |\rho-r| \leq \frac{1-r}{3} \\ |\theta-\varphi| \leq \frac{1-r}{3}}} \left(\log \left| \frac{1 - r\rho e^{i(\varphi-\theta)}}{re^{i\varphi} - \rho e^{i\theta}} \right| \right)^p \mu^{p-1} \left(\square(re^{i\theta}, \frac{2}{3}(1-r)) \right) d\mu(\rho e^{i\theta}) d\varphi = \\
&\leq 2 \int_{||\zeta|-r| \leq \frac{1}{3}(1-r)} \mu^{p-1} \left(\square(re^{i \arg \zeta}, \frac{2}{3}(1-r)) \right) \int_{-\pi}^{\pi} \left(\log \left| \frac{1 - re^{i\varphi} \bar{\zeta}}{re^{i\varphi} - \zeta} \right| \right)^p d\varphi d\mu(\zeta).
\end{aligned}$$

We know that ([6]) for any $a, b \in \mathbb{C}$, and $p > 1$

$$\int_{-\pi}^{\pi} \left| \log \left| \frac{a - e^{i\theta}}{b - e^{i\theta}} \right| \right|^p d\theta \leq C(p) |a - b|$$

holds. Using this inequality we obtain ($r \in (\frac{1}{2}, 1)$)

$$I_1 \leq 4C(p)(1-r) \int_{||\zeta|-r| \leq \frac{1}{3}(1-r)} \mu^{p-1} \left(\square(re^{i \arg \zeta}, \frac{2}{3}(1-r)) \right) d\mu(\zeta). \quad (30)$$

In order to proceed we need the following lemma.

Lemma 1. *Let ν be a 2π periodic positive Borel measure on \mathbb{R} , $p \geq 1$, $\delta \in (0, \pi)$. Then*

$$\int_{[-\pi, \pi)} \nu^{p-1}((\theta - \delta, \theta + \delta)) d\nu(\theta) \leq \frac{2^{p+1}}{\delta} \int_{[-\pi, \pi)} \nu^p((\theta - \delta, \theta + \delta)) d\theta. \quad (31)$$

Proof of Lemma 1. First, we prove (31) for $p = 1$ ¹.

We have

$$\begin{aligned}
\int_{[-\pi, \pi)} d\nu(\theta) &= \int_{[-\pi, \pi)} \frac{1}{\delta} \int_{\theta - \frac{\delta}{2}}^{\theta + \frac{\delta}{2}} dx d\nu(\theta) \leq \int_{[-\pi - \frac{\delta}{2}, \pi + \frac{\delta}{2})} dx \int_{[x - \frac{\delta}{2}, x + \frac{\delta}{2})} \frac{1}{\delta} d\nu(\theta) = \\
&= \int_{[-\pi - \frac{\delta}{2}, \pi + \frac{\delta}{2})} \frac{\nu([x - \frac{\delta}{2}, x + \frac{\delta}{2}))}{\delta} dx \leq 2 \int_{[-\pi, \pi)} \frac{\nu([x - \frac{\delta}{2}, x + \frac{\delta}{2}))}{\delta} dx \leq \quad (32)
\end{aligned}$$

$$\leq 2 \int_{[-\pi, \pi)} \frac{\nu((x - \delta, x + \delta))}{\delta} dx. \quad (33)$$

We now consider arbitrary finite $p > 1$. Applying (32) with $d\nu_1(\theta) =$

¹The author thanks Prof. Sergii Favorov for the idea of the proof of this lemma.

$\nu^{p-1}((\theta - \delta, \theta + \delta))d\nu(\theta)$, we get

$$\begin{aligned}
\int_{[-\pi, \pi)} \nu^{p-1}((\theta - \delta, \theta + \delta))d\nu(\theta) &= \int_{[-\pi, \pi)} d\nu_1(\theta) \leq 2 \int_{[-\pi, \pi)} \frac{\nu_1([x - \frac{\delta}{2}, x + \frac{\delta}{2}))}{\delta} dx = \\
&= 2 \int_{[-\pi, \pi)} \int_{[x - \frac{\delta}{2}, x + \frac{\delta}{2})} \nu^{p-1}((\theta - \delta, \theta + \delta))d\nu(\theta) dx \leq \\
&\leq 2 \int_{[-\pi, \pi)} \frac{\nu^{p-1}((x - \frac{3\delta}{2}, x + \frac{3\delta}{2}))\nu([x - \frac{\delta}{2}, x + \frac{\delta}{2}))}{\delta} dx \leq \\
&\leq 2 \int_{[-\pi, \pi)} \frac{\nu^p((x - \frac{3\delta}{2}, x + \frac{3\delta}{2}))}{\delta} dx \leq \\
&\leq 2 \int_{[-\pi, \pi)} \frac{\nu^p((x - \frac{3\delta}{2}, x) \cup [x, x + \frac{3\delta}{2}))}{\delta} dx \leq \\
&\leq 2^p \int_{[-\pi, \pi)} \frac{\nu^p((x - \frac{3\delta}{2}, x))}{\delta} dx + 2^p \int_{[-\pi, \pi)} \frac{\nu^p([x, x + \frac{3\delta}{2}, x))}{\delta} dx \leq \\
&\leq 2^{p+1} \int_{[-\pi, \pi)} \frac{\nu^p((x - \delta, x - \delta))}{\delta} dx.
\end{aligned}$$

The lemma is proved. \square

Let us continue the proof of the sufficiency.

We denote the nondecreasing function

$$N_r(\theta) = \lambda(\{\rho e^{i\alpha} : |r - \rho| \leq \frac{2}{3}(1 - r), -\pi \leq \alpha \leq \theta\}), \quad \theta \in [-\pi, \pi).$$

We extend it on the real axis preserving monotonicity by $N_r(x + 2\pi) - N_r(x) = N_r(2\pi) - N_r(0)$, $x \in \mathbb{R}$. Let ν_r be the corresponding Stieltjes measure on \mathbb{R} . Estimate (30) can be written in the form

$$\begin{aligned}
I_1 &\leq \frac{C}{(1 - r)^{p-1}} \int_{\|\zeta| - r| \leq \frac{1}{3}(1 - r)} \lambda^{p-1}\left(\square(re^{i \arg \zeta}, \frac{2}{3}(1 - r))\right) d\lambda(\zeta) = \\
&= \frac{C}{(1 - r)^{p-1}} \int_{-\pi}^{\pi} \nu_r^{p-1}\left(\left[\theta - \frac{2}{3}(1 - r), \theta + \frac{2}{3}(1 - r)\right]\right) d\nu_r(\theta) \leq \\
&\leq 2^{p+1} \frac{3C}{2(1 - r)^p} \int_{-\pi}^{\pi} \nu_r^p\left(\left[\theta - \frac{2}{3}(1 - r), \theta + \frac{2}{3}(1 - r)\right]\right) d\theta \leq \\
&\leq \frac{C(p)}{(1 - r)^p} \int_{-\pi}^{\pi} \lambda^p\left(\square(re^{i\theta}, \frac{2}{3}(1 - r))\right) d\theta \leq \frac{C}{(1 - r)^p} (1 - r)^{p\gamma}.
\end{aligned}$$

We have used Lemma 1 and the assumption of the theorem on the complete measure.

Thus, we have

$$\left(\int_{-\pi}^{\pi} |u_1(re^{i\varphi})|^p d\varphi\right)^{\frac{1}{p}} \leq C(p)(1 - r)^{\gamma-1}. \quad (34)$$

Let us estimate $u_2(z) = -\int_{\mathbb{D}} K(z, \zeta) d\tilde{\lambda}(\zeta)$, where $d\tilde{\lambda}(\zeta) = \chi_{\mathbb{D} \setminus D^*(z, \frac{1}{r})}(\zeta) d\lambda(\zeta)$.

Since $\text{supp } \tilde{\lambda} \cap D^*(z, \frac{1}{r}) = \emptyset$, for $\zeta \notin D^*(z, \frac{1}{r})$ we have by Proposition 1 that

$$K(z, \zeta) \leq \frac{49(1 - |z|^2)}{|1 - z\bar{\zeta}|^2}. \quad (35)$$

Let $E_n = E_n(re^{i\varphi}) = \mathcal{C}(\varphi, 2^n(1-r))$, $n \in \mathbb{N}$, $E_0 = \emptyset$. Then for $\zeta = \rho e^{it} \in \mathbb{D} \setminus E_n(z)$, $n \geq 1$, we have

$$|1 - \rho re^{i(\varphi-t)}| \geq |1 - \rho e^{i(\varphi-t)}| - \rho(1-r) \geq 2^n(1-r) - (1-r) \geq 2^{n-1}(1-r).$$

and $|1 - \rho re^{i(\varphi-t)}| \geq 1 - r\rho \geq 1 - r$ for $\zeta \in E_1(z)$. Therefore $(\frac{1}{p} + \frac{1}{p'} = 1)$

$$\begin{aligned} |u_2(re^{i\varphi})|^p &\leq \left(\left(\sum_{n=1}^{\lfloor \log_2 \frac{1}{1-r} \rfloor} \int_{E_{n+1} \setminus E_n} + \int_{E_1} \right) \frac{49(1-r^2)}{|1 - re^{i\varphi}\zeta|^2} d\tilde{\lambda}(\zeta) \right)^p \leq \\ &\leq 49^p \left(\sum_{n=1}^{\lfloor \log_2 \frac{1}{1-r} \rfloor} \int_{E_{n+1} \setminus E_n} \frac{2(1-r)}{(2^{n-1}(1-r))^2} d\tilde{\lambda}(\zeta) + \int_{E_1} \frac{2}{1-r} d\tilde{\lambda}(\zeta) \right)^p < \\ &\leq \left(\frac{400}{1-r} \right)^p \sum_{n=1}^{\lfloor \log_2 \frac{1}{1-r} \rfloor + 1} \frac{\tilde{\lambda}^p(E_n(z))}{2^{\frac{3}{2}np}} \left(\sum_{n=1}^{\infty} \frac{1}{2^{\frac{np'}{2}}} \right)^{\frac{p}{p'}} \leq \frac{C(p)}{(1-r)^p} \sum_{n=1}^{\infty} \frac{\tilde{\lambda}^p(E_n(z))}{2^{\frac{3}{2}np}}. \end{aligned} \quad (36)$$

It follows from the latter inequalities and the assumption of the theorem that $(r \in [\frac{1}{2}, 1))$

$$\begin{aligned} \int_0^{2\pi} |u_2(re^{i\varphi})|^p d\varphi &\leq \frac{C(p)}{(1-r)^p} \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\tilde{\lambda}^p(E_n(re^{i\varphi}))}{2^{\frac{3}{2}np}} d\varphi \leq \\ &\leq \frac{C(p)}{(1-r)^p} \sum_{n=1}^{\infty} \frac{(2^n(1-r))^{p\gamma}}{2^{\frac{3}{2}np}} = \frac{C(p)}{(1-r)^{p(1-\gamma)}} \sum_{n=1}^{\infty} 2^{np(\gamma - \frac{3}{2})} = \frac{C(p, \gamma)}{(1-r)^{p(1-\gamma)}}. \\ \left(\int_0^{2\pi} |u_2(re^{i\varphi})|^p d\varphi \right)^{\frac{1}{p}} &\leq \frac{C(\gamma, p)}{(1-r)^{1-\gamma}}, \quad r \in [0, 1). \end{aligned}$$

Sufficiency of Theorem 1 is proved.

Necessity. Using property d) of Proposition 1, we obtain

$$|u(re^{i\theta})| \geq \int_{\mathcal{C}(\varphi, \frac{1-r}{2})} K(re^{i\varphi}, \zeta) d\lambda(\zeta) \geq \frac{1}{12} \int_{\mathcal{C}(\varphi, \frac{1-r}{2})} \frac{1-r^2}{|re^{i\varphi} - \zeta|^2} d\lambda(\zeta).$$

Elementary geometric arguments show that $|re^{i\varphi} - \rho e^{i\theta}| \leq |re^{i\varphi} - e^{i\theta}|$ for $1 > \rho \geq r \geq 0$. It then follows that

$$\begin{aligned} |u(re^{i\theta})| &\geq \frac{1}{12} \int_{\mathcal{C}(\varphi, \frac{1-r}{2})} \frac{1-r^2}{|re^{i\varphi} - e^{i\theta}|^2} d\lambda(\rho e^{i\theta}) \geq \\ &\geq \frac{1}{3(\frac{\pi^2}{4} + 1)} \frac{1-r^2}{(1-r)^2} \int_{\mathcal{C}(\varphi, \frac{1-r}{2})} d\lambda(\rho e^{i\theta}) \geq \frac{\lambda(\mathcal{C}(\varphi, \frac{1-r}{2}))}{3(\frac{\pi^2}{4} + 1)(1-r)}. \end{aligned}$$

By the assumption of the theorem we deduce that

$$\frac{C}{(1-r)^{(1-\gamma)p}} \geq \int_0^{2\pi} |u(re^{i\varphi})|^p d\varphi \geq C \frac{\int_0^{2\pi} \lambda^p(\mathcal{C}(\varphi, \frac{1-r}{2})) d\varphi}{(1-r)^p}.$$

Hence $\int_0^{2\pi} \lambda^p(\mathcal{C}(\varphi, \frac{1-r}{2})) d\varphi = O((1-r)^{\gamma p})$ as $r \uparrow 1$. This completes the proof of necessity. \square

Proof of Theorem 3. Due to (7) we write

$$\begin{aligned} u(z) &= - \int_{\mathbb{D}} K_s(z, \zeta) d\lambda(\zeta) = \\ &= - \int_{D^*(z, \frac{1}{7})} K_s(z, \zeta) d\lambda(\zeta) - \int_{\mathbb{D} \setminus D^*(z, \frac{1}{7})} K_s(z, \zeta) d\lambda(\zeta) \equiv u_1 + u_2. \end{aligned}$$

According to (24)

$$|u_1(z)| \leq C(s) \int_{D^*(z, \frac{1}{7})} \log \left| \frac{1 - z\bar{\zeta}}{z - \zeta} \right| d\mu(\zeta).$$

Its estimate repeats that for the case $s = 0$.

Let us estimate p th means of $u_2(z)$. Using Proposition 2 we deduce (cf. proof of Theorem 1)

$$\begin{aligned} |u_2(re^{i\varphi})|^p &\leq \left(\left(\sum_{n=1}^{\lfloor \log_2 \frac{1}{1-r} \rfloor} \int_{E_{n+1} \setminus E_n} + \int_{E_1} \right) \frac{C(s)}{|1 - re^{i\varphi}\bar{\zeta}|^{s+1}} |d\tilde{\lambda}(\zeta)| \right)^p \leq \\ &\leq \frac{C}{(1-r)^{(s+1)p}} \sum_{n=1}^{\lfloor \log_2 \frac{1}{1-r} \rfloor + 1} \frac{|\tilde{\lambda}|^p(E_n(z))}{2^{(s+\frac{1}{2})np}} \left(\sum_{n=1}^{\infty} \frac{1}{2^{\frac{np'}{2}}} \right)^{\frac{p}{p'}} \leq \\ &\leq \frac{C}{(1-r)^{(s+1)p}} \sum_{n=1}^{\infty} \frac{|\tilde{\lambda}|^p(E_n(z))}{2^{(s+\frac{1}{2})np}}. \end{aligned}$$

It follows from the latter inequalities and the assumption of the theorem that

$$\begin{aligned} \int_0^{2\pi} |u_2(re^{i\varphi})|^p d\varphi &\leq \frac{C(p)}{(1-r)^{(s+1)p}} \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{|\tilde{\lambda}|(E_n(re^{i\varphi}))}{2^{(s+\frac{1}{2})np}} d\varphi \leq \\ &\leq \frac{C(p)}{(1-r)^{(s+1)p}} \sum_{n=1}^{\infty} \frac{(2^n(1-r))^{p\gamma}}{2^{(s+\frac{1}{2})np}} = \frac{C(p, \gamma)}{(1-r)^{p(s+1-\gamma)}}, \quad r \in \left[\frac{1}{2}, 1 \right). \end{aligned}$$

Finally,

$$\left(\int_0^{2\pi} |u_2(re^{i\varphi})|^p d\varphi \right)^{\frac{1}{p}} \leq \frac{C(\gamma, p)}{(1-r)^{s+1-\gamma}}, \quad r \in \left[\frac{1}{2}, 1 \right).$$

\square

Proof of Theorem 3. Without loss of generality we assume that $\text{supp } \mu_u \subset \{z \in \mathbb{D} : |1 - z| < 2(1 - |z|)\} =: \Delta$.

Necessity. Note that $\mathcal{R}(1 - \delta, 2\delta) \subset \mathcal{C}(\varphi, \delta)$ for $\varphi \in [-\delta, \delta]$. Applying Theorem 1 we obtain

$$\left(\int_{-\delta}^{\delta} \lambda^p(\mathcal{R}(1 - \delta, 2\delta)) d\varphi \right)^{\frac{1}{p}} = O(\delta^{1-\alpha}), \quad 0 < \delta < 1,$$

or

$$\mu_u(\mathcal{R}(1 - \delta, 2\delta)) = O(\delta^{-\alpha - \frac{1}{p}}), \quad 0 < \delta < 1.$$

Since

$$\Delta \subset \overline{D(0, \frac{1}{2})} \cup \bigcup_{n=1}^{\infty} \mathcal{R}(1 - 2^{-n}, 2^{1-n}) \quad (37)$$

we deduce

$$n(1 - 2^{-k}, u) \leq C \sum_{n=1}^k 2^{n(\alpha + \frac{1}{p})} + C = O(2^{k(\alpha + \frac{1}{p})}), \quad k \in \mathbb{N},$$

and the assertion follows.

Sufficiency. It follows from the assumptions that

$$\lambda(\mathcal{R}((1 - \delta)e^{i\varphi}, 4\delta)) = O(\delta^{1-\alpha-\frac{1}{p}}), \quad \delta \downarrow 0.$$

Then

$$\lambda(\mathcal{C}(\varphi, \delta)) \leq \lambda\left(\bigcup_{n=0}^{\infty} \mathcal{R}(1 - \frac{\delta}{2^n} e^{i\varphi}, \frac{4\delta}{2^n})\right) \leq C \sum_{n=0}^{\infty} \left(\frac{\delta}{2^n}\right)^{1-\alpha-\frac{1}{p}} = O(\delta^{1-\alpha-\frac{1}{p}}), \delta \downarrow 0.$$

Since $\text{supp } \mu_u \subset \Delta$, we have

$$\int_{-\pi}^{\pi} \lambda^p(\mathcal{C}(\varphi, \delta)) d\varphi = \int_{-2\pi\delta}^{2\pi\delta} \lambda^p(\mathcal{C}(\varphi, \delta)) d\varphi = O(\delta \delta^{p(1-\alpha-\frac{1}{p})}) = O(\delta^{p(1-\alpha)}), \delta \downarrow 0.$$

It remains to apply Theorem 1. \square

Proof of Theorem 6. We confine ourselves to the case $s = 0$. We keep the notation from the proof of Theorem 1. It follows from estimate (30) that

$$\int_{-\pi}^{\pi} |u_1(re^{i\varphi})|^p d\varphi \leq (1-r)n^{p-1} \left((r + \frac{2}{3}(1-r), u) \right) n \left((r + \frac{1}{2}(1-r), u) \right) = O((1-r)^{-\alpha p}). \quad (38)$$

Let us estimate p th mean of u_2 . We use estimate (35), integral Minkowski's inequality ([25, §A1]), standard estimates, and integration by parts

$$\begin{aligned} & \left(\int_{-\pi}^{\pi} |u_2(re^{i\varphi})|^p d\varphi \right)^{\frac{1}{p}} \leq C \left(\int_{-\pi}^{\pi} \left(\int_{\mathbb{D}} \frac{1-r^2}{|1-re^{i\varphi}\bar{\zeta}|^2} d\lambda(\zeta) \right)^p d\varphi \right)^{\frac{1}{p}} \leq \\ & \leq C \int_{\mathbb{D}} \left(\int_{-\pi}^{\pi} \left(\frac{1-r^2}{|1-re^{i\varphi}\bar{\zeta}|^2} \right)^p d\varphi \right)^{\frac{1}{p}} d\lambda(\zeta) \leq C \int_{\mathbb{D}} \frac{1-r}{(1-r|\zeta|)^{2-\frac{1}{p}}} d\lambda(\zeta) = \\ & = C(1-r) \int_0^1 \frac{(1-t)dn(t, u)}{(1-rt)^{2-\frac{1}{p}}} \leq C(1-r) \left(\int_0^r \frac{dn(t, u)}{(1-t)^{1-\frac{1}{p}}} + \right. \\ & \left. + \int_r^1 \frac{(1-t)dn(t, u)}{(1-r)^{2-\frac{1}{p}}} \right) \leq \frac{C}{(1-r)^{1-\frac{1}{p}}} \int_r^1 n(t, u) dt = O((1-r)^{-\alpha}), \quad r \uparrow 1. \end{aligned}$$

Taking into account (38), we obtain desired estimate. \square

4 Examples

Example 1. Following Linden [21, Lemma 1], given $\alpha \geq 1$, $\beta \in [0, 1]$, we consider the sequence of complex numbers

$$a_{k,m} = (1 - 2^{-k})e^{im2^{-k}}, \quad 1 \leq m \leq [2^{k\beta}] \quad (39)$$

where each of numbers (39) is counted $[2^{\alpha k}]$ times. Then for $\mathcal{P}(z) = \mathcal{P}(z, (a_{k,m}), s)$, where $s = \min\{q \in \mathbb{N} : q > \alpha + \beta - 1\}$ we have (see [21]) $n(r, \mathcal{P}) \asymp \left(\frac{1}{1-r}\right)^{\alpha+\beta}$, $\nu(r, \mathcal{P}) \asymp \left(\frac{1}{1-r}\right)^\alpha$, $r \uparrow 1$. Therefore, by Theorem A [18] $\rho_M[\mathcal{P}] = \alpha$. In [21] it is proved that $\rho_p[\log |\mathcal{P}|] = \alpha + \frac{\beta-1}{p}$. We are going to prove that

$$\left(\int_0^{2\pi} \lambda^p(\mathcal{C}(\varphi, \delta)) d\varphi \right)^{\frac{1}{p}} \geq C(\delta^{s+1-\alpha-\frac{\beta-1}{p}}), \quad \delta \downarrow 0. \quad (40)$$

It would imply that restriction (18) could not be weakened.

We first assume that $\beta \in (0, 1)$. Given $\delta \in (0, \delta_0)$ we define $\varphi_\delta = \delta^{1-\beta} - \pi\delta$, where δ_0 is chosen such that $\varphi_\delta > 0$. Note that $\varphi_\delta \sim \delta^{1-\beta}$, $\delta \downarrow 0$. According to the definition of $\mathcal{C}(\varphi, \delta)$, $a_{k,m} \in \mathcal{C}(\varphi, \delta)$ if and only if

$$1 - |a_{k,m}| = 2^{-k} \leq \delta, \quad \varphi - \pi\delta \leq m2^{-k} \leq \varphi + \pi\delta. \quad (41)$$

Let $G(\varphi, \delta)$ denote the set of (k, m) such that (41) is valid. It is easy to check that for $\varphi \in (0, \varphi_\delta)$ the set $G(\varphi, \delta)$ is not empty. Let

$$k_1(\varphi) = \min\{k : 2^{-k}[2^{\beta k}] \leq \varphi + \pi\delta\},$$

where $\varphi \in (0, \varphi_\delta)$. Since $k_1(\varphi)$ tends to infinity uniformly with respect to $\varphi \in (0, \varphi_\delta)$ as $\delta \downarrow 0$, one can choose δ_1 so small that for all $\delta \in (0, \delta_1)$, $\varphi \in (0, \varphi_\delta)$ and $k \geq k_1(\varphi)$ the inequality $\frac{2^{-\beta k}}{(1-\beta)(1-2^{\beta k})\log 2} \leq 1$ holds. Under this assumptions we deduce subsequently from the definition of $k_1 = k_1(\varphi)$ that

$$\begin{aligned} |2^{k_1}(\varphi + \pi\delta) - 2^{\beta k_1}| &< 1, \\ \frac{1 - 2^{-\beta k_1}}{\varphi + \pi\delta} &< 2^{k_1(1-\beta)} < \frac{1 + 2^{-\beta k_1}}{\varphi + \pi\delta}, \\ |k_1 - \frac{1}{1-\beta} \log_2 \frac{1}{\varphi + \pi\delta}| &< 1. \end{aligned} \quad (42)$$

It follows from the definition of φ_δ and (42) that

$$2^{k_1} > \frac{(1 - 2^{-k_1\beta})^{\frac{1}{1-\beta}}}{\delta} > \frac{2}{\pi\delta}, \quad 0 < \varphi < \varphi_\delta, \quad (43)$$

Then, according to (41), (43) for $\delta \in (0, \min\{\delta_0, \delta_1\})$ and $\varphi \in (\frac{1}{2}\varphi_\delta, \varphi_\delta)$, $\delta \downarrow 0$

$$\begin{aligned} \lambda(\mathcal{C}(\varphi, \delta)) &= \sum_{(k,m) \in G(\varphi, \delta)} [2^{\alpha k}] 2^{-k(s+1)} \geq \sum_{m=[2^{k_1}(\varphi - \pi\delta)]+1}^{[2^{k_1}(\varphi + \pi\delta)]} [2^{\alpha k_1}] 2^{-k_1(s+1)} \geq \\ &\geq [2^{\alpha k_1}] 2^{-k_1(s+1)} (2^{k_1} 2\pi\delta - 2) \geq [2^{\alpha k_1}] 2^{-k_1 s} \pi\delta \geq \frac{\pi\delta}{2} 2^{(\alpha-s)k_1} \sim \frac{\pi\delta}{2} \left(\frac{1}{\varphi}\right)^{\frac{\alpha-s}{1-\beta}}. \end{aligned} \quad (44)$$

It follows from the last estimate that

$$\begin{aligned} \left(\int_0^{2\pi} (\lambda(C(\varphi, \delta)))^p d\varphi \right)^{\frac{1}{p}} &\geq \frac{\pi\delta}{2} \left(\int_{\varphi_{\delta/2}}^{\varphi_{\delta}} (\varphi^{\frac{s-\alpha}{1-\beta}})^p d\varphi \right)^{\frac{1}{p}} = \\ &= \frac{\pi}{2(\frac{s-\alpha}{1-\beta}p+1)} \delta \varphi^{\frac{s-\alpha}{1-\beta}+\frac{1}{p}} \Big|_{\varphi_{\delta/2}}^{\varphi_{\delta}} \sim C(s, \alpha, p) \delta^{1+s-\alpha-\frac{\beta-1}{p}}, \quad \delta \downarrow 0. \end{aligned}$$

In the case $\beta = 1$ the arguments could be simplified. By the choice of s , $s > \alpha$. For $0 < \varphi \leq \frac{1}{2}$, according to (41) we deduce

$$\begin{aligned} \lambda(C(\varphi, \delta)) &= \sum_{k=\lfloor \log_2 \frac{1}{\delta} \rfloor + 1}^{\infty} \sum_{m=\lfloor 2^k(\varphi-\pi\delta) \rfloor + 1}^{2^k(\varphi+\pi\delta)} [2^{\alpha k}] 2^{-k(s+1)} \geq \\ &\geq \sum_{k=\lfloor \log_2 \frac{1}{\delta} \rfloor + 1}^{\infty} [2^{\alpha k}] 2^{-k(s+1)} (2^k 2\pi\delta - 2) \geq \sum_{k=\lfloor \log_2 \frac{1}{\delta} \rfloor + 1}^{\infty} [2^{\alpha k}] 2^{-ks} \pi\delta \geq \\ &\geq \frac{\pi\delta}{2} \sum_{k=\lfloor \log_2 \frac{1}{\delta} \rfloor + 1}^{\infty} 2^{(\alpha-s)k} \asymp \delta^{1+s-\alpha}. \end{aligned}$$

Hence,

$$\left(\int_0^{2\pi} (\lambda(C(\varphi, \delta)))^p d\varphi \right)^{\frac{1}{p}} \geq C(s, \alpha, p) \delta^{1+s-\alpha}, \quad \delta \downarrow 0.$$

If $\beta = 0$, then all zeros $a_k = 1 - 2^{-k}$ are located on $[0, 1)$, and $s > \alpha - 1$. For $\varphi \in (-\pi\delta, \pi\delta)$ we then have according to (41)

$$\lambda(C(\varphi, \delta)) = \sum_{k=\lfloor \log_2 \frac{1}{\delta} \rfloor + 1}^{\infty} [2^{\alpha k}] 2^{-k(s+1)} \geq C \sum_{k=\lfloor \log_2 \frac{1}{\delta} \rfloor + 1}^{\infty} 2^{(\alpha-s-1)k} \asymp \delta^{1+s-\alpha}.$$

Then

$$\left(\int_0^{2\pi} (\lambda(C(\varphi, \delta)))^p d\varphi \right)^{\frac{1}{p}} \geq C \delta^{1+s-\alpha} \left(\int_{-\pi\delta}^{\pi\delta} d\varphi \right)^{\frac{1}{p}} = C(s, \alpha, p) \delta^{1+s-\alpha+\frac{1}{p}}, \quad \delta \downarrow 0$$

as required.

Example 2. Let $f(z) = \exp\left\{\left(\frac{1}{1-z}\right)^{q+1}\right\}$, $q > -1$, $f(0) = e$. In this case $f(z)$ is of the form (20) with $(a_k) = \emptyset$, $\psi^*(\theta) = H(\theta)m_0$, $m_0 > 0$, where $H(\theta)$ the Heaviside function, i.e. $\lambda(\zeta) = m_0\delta(\zeta - 1)$. It is easy to check that

$$\left(\int_0^{2\pi} (\lambda(C(\varphi, \delta)))^p d\varphi \right)^{\frac{1}{p}} = m_0 (2\pi\delta)^{\frac{1}{p}},$$

and $m_p(r, \log |f|) \asymp (1-r)^{\frac{1}{p}-q-1}$.

References

- [1] M. Brelo. On topologies and boundaries in potential theory, Springer-Verlag, Berlin-Heidelberg-New York, 1971, Lect.Notes in Mathematics, V.175.

- [2] J. Brune, J. Otrega-Cerdá, On L^p -solutions of the Laplace equation and zeros of holomorphic functions, *Annali della Scuola Normale Superiore di Pisa – Classe di Scienze* **24** (1997), no. 3, 571–591.
- [3] I. Chyzhykov, A generalization of Hardy-Littlewood's theorem, *Math. Methods and Physicomechanical Fields*, **49** (2006), no.2, 74–79 (in Ukrainain)
- [4] I. E. Chyzhykov. Growth of analytic functions in the unit disc and complete measure in the sense of Grishin, *Mat. Stud.* **29** (2008), no. 1, 35–44.
- [5] I. Chyzhykov, Argument of bounded analytic functions and Frostman's type conditions, *Ill. J. Math.* **2** (2009), no.2, 515–531.
- [6] I. Chyzhykov, Zero distribution and factorisation of analytic functions of slow growth in the unit disc, *Proc. Amer. Math. Soc.*, **141** (2013), 1297–1311.
- [7] I. Chyzhykov, S. Skaskiv, Growth, zero distribution and factorization of analytic functions of moderate growth in the unit disc, Blaschke products and their applications, *Fields Inst. Comm.*, **65** (2013), 159–173.
- [8] E.F.Collingwood, A.J.Lohwater, The theory of cluster sets, Cambridge Univer. Press, 1966.
- [9] Djrbashian M.M. Integral transforms and representations of functions in the complex domain, Moscow, Nauka, 1966 (in Russian).
- [10] M. M. Djrbashian, *Theory of factorization and boundary properties of functions meromorphic in the disc*, Proceedings of the ICM, Vancouver, BC, 1974.
- [11] P. L. Duren, Theory of H^p spaces, Academic press, NY and London, 1970. – 258 pp.
- [12] S. J. Gardiner, Growth properties of p th means of potentials in the unit ball, *Proc. Amer. Math. Soc.* **103** (1988) 861–869.
- [13] A.F. Grishin, On growth regularity of subharmonic functions, II, Theory of functions, func. anal. and appl. (1968), no.7, 59–84 (in Russian)
- [14] A.Grishin, Continuity and asymptotic continuity of subharmonic functions, *Math. Physics, Analysis, Geometry, ILPTE* **1** (1994), no.2, 193–215 (in Russian).
- [15] M.A. Fedorov, A.F.Grishin, Some questions of the Nevanlinna theory for the complex half-plane, *Math. Physics, Analysis and Geometry (Kluwer Acad. Publish.)* **1** (1998), no.3, 223–271.
- [16] W.K.Hayman, Meromorphic functions, Oxford, Clarendon press, 1964.
- [17] W.K.Hayman, P.B.Kennedy, Subharmonic functions, V.1. Academic press, London-New York-San Francisco, 1976.
- [18] Linden C.N. The representation of regular functions, *J. London Math. Soc.* **39** (1964), 19–30.

- [19] C.N. Linden, On Blaschke products of restricted growth, Pacific J. of Math., **38** (1971) , no.2, 501–513.
- [20] C.N. Linden, Integral logarithmic means for regular functions, Pacific J. of Math., **138** (1989) , no.1, 119–127.
- [21] C.N. Linden, The characterization of orders for regular functions, Math. Proc. Cambodge Phil. Soc. 111 (1992), no.2, 299–307.
- [22] G.R. MacLane, L.A. Rubel, On the growth of the Blaschke products, Canad. J. Math. **21** (1969), 595–600.
- [23] Mykytyuk, Ya. V., Vasyl'kiv, Ya. V., The boundedness criteria of integral means of Blaschke product logarithms, Dopov. Nats. Akad. Nauk Ukr., Mat. Prirodozn Tekh. Nauki **8** (2000), 10–14. (in Ukrainian)
- [24] A.V.Rybkin, Convergence of arguments of Blaschke products in L_p -metrics, Proc. Amer. Math. Soc. **111** (1991), no.3, 701–708.
- [25] Stein E.M. Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, New Jersey, 1970.
- [26] Stoll M. On the rate of growth of the means M_p of holomorphic and pluri-harmonic functions on the ball, J.Math. Anal. Appl. **93** (1983), 109–127.
- [27] Stoll M., Rate of growth of p th means of invariant potentials in the unit ball of \mathbb{C}^n , J. Math. Anal. Appl. **143** (1989), 480–499.
- [28] Tsuji M. Potential theory in modern function theory. – Chelsea Publishing Co. Reprinting of the 1959 edition. New York, 1975.